arXiv:1502.00708v1 [math.CO] 3 Feb 2015

VIZING'S CONJECTURE FOR ALMOST ALL PAIRS OF GRAPHS

AZIZ CONTRACTOR AND ELLIOT KROP

Abstract. For any graph $G = (V, E)$, a subset $S \subseteq V$ *dominates* G if all vertices are contained in the closed neighborhood of S, that is $N[S] = V$. The minimum cardinality over all such S is called the domination number, written $\gamma(G)$. In 1963, V.G. Vizing conjectured that $\gamma(G \square H) \geq \gamma(G)\gamma(H)$ where $\square$ stands for the Cartesian product of graphs. In this note, we prove that if $|G| \geq \gamma(G)\gamma(H)$ and $|H| \geq \gamma(G)\gamma(H)$, then the conjecture holds. This result quickly implies Vizing's conjecture for almost all pairs of graphs G, H with $|G| \geq |H|$, satisfying $|G| \leq q^{\frac{|H|}{\log_q |H|}}$ for $q = \frac{1}{1-p}$ and p the edge probability of the Erdős-Rényi random graph.

2010 Mathematics Subject Classification: 05C69

Keywords: Domination number, Cartesian product of graphs, Vizing's conjecture

1. Introduction

For basic graph theoretic notation and definitions see Diestel [5]. All graphs $G(V, E)$ are finite, simple, undirected graphs with vertex set V and edge set E. We may refer to the vertex set and edge set of G as $V(G)$ and $E(G)$, respectively.

For any graph $G = (V, E)$, a subset $S \subseteq V$ *dominates* G if $N[S] = G$. The minimum cardinality of $S \subseteq V$, so that S dominates G is called the *domination number* of G and is denoted $\gamma(G)$.

Definition 1.1. *The* Cartesian product *of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$, denoted by $G_1 \square G_2$, is a graph with vertex set $V_1 \times V_2$ and edge set $E(G_1 \square G_2) = \{((u_1, v_1), (u_2, v_2)) : v_1 = v_2$ and $(u_1, u_2) \in E_1,$ or $u_1 = u_2$ and $(v_1, v_2) \in E_2\}$.*

The famous conjecture of Vadim G. Vizing (1963) [9] states that

$$\gamma(G \square H) \geq \gamma(G)\gamma(H). \tag{1.1}$$

Previous work on this problem has been reviewed in the excellent survey [3].

One of the earliest significant results is that of Barcalkin and German [1], who showed that the conjecture holds for decomposable graphs, that is, graphs G with vertex sets which can be disjointly covered by $\gamma(G)$ cliques.

A generalization of those techniques came much later in [4]. The authors defined the related parameter of fair domination and showed that graphs

Date: February 4, 2015.

with identical fair domination number and domination number satisfy the conjecture. However, finding bounds on fair domination numbers has been diffcult so far.

The best current bound for the conjectured inequality was shown in 2010 by Suen and Tarr [8],

$$\gamma(G\Box H) \geq \frac{1}{2}\gamma(G)\gamma(H) + \frac{1}{2}\min\{\gamma(G), \gamma(H)\}.$$

We take a different point of view and show that for any two fixed domination numbers, γ_1 and γ_2, all graphs attaining those domination numbers, respectively, with orders larger than the product $\gamma_1\gamma_2$, satisfy the conjecture. The proof of this fact is an elementary counting argument. By applying a result of Dryer [6] it is easy to show that Vizing's conjecture holds for almost all graphs G, H with $|G| \geq |H|$ satisfying the order bound condition $|G| \leq q^{\frac{|H|}{\log_q |H|}}$ for $q = \frac{1}{1-p}$ and p the edge probability of the Erdős-Rényi random graph.

2. Counting Vertices in Blocks

Given vertex partitions of G into sets $G_1, \ldots, G_k$ and H into sets $H_1, \ldots, H_l$, a *block* of $G\Box H$ is the induced subgraph $G_i\Box H_j$, for some i, $1 \leq i \leq k$, and j, $1 \leq j \leq l$.

Theorem 2.1. *For every two graphs G and H satisfying $|G| \geq \gamma(G)\gamma(H)$ and $|H| \geq \gamma(G)\gamma(H)$, $\gamma(G\Box H) \geq \gamma(G)\gamma(H)$.*

Proof. Let D be a minimum dominating set of $G\Box H$. Suppose $\gamma(G) = k$ and $\gamma(H) = l$. Partition the vertices of G arbitrarily into sets $G_1, \ldots, G_k$ and the vertices of H into sets $H_1, \ldots, H_l$ so that for any i, $1 \leq i \leq k$, and any j, $1 \leq j \leq l$, $|G_i| \geq \gamma(H)$ and $|H_j| \geq \gamma(G)$. Furthermore, we call a block $B_{i,j} = G_i\Box H_j$ a *G-cell block* if there are at least $|H_j|$ vertices of D in $G\Box H_j$. We say B is a *H-cell block* if there are at least $|G_i|$ vertices of D in $G_i\Box H$.

Observation 2.2. *Every block is either a G-cell block or an H-cell block.*

Without loss of generality, suppose $\gamma(G) \geq \gamma(H)$. If no block $\{B_{1,1}, B_{2,1}, \ldots, B_{k,1}\}$ is a G-cell block, then each is an H-cell block and we count at least $\gamma(G)\gamma(H)$ vertices of D. Thus, we can find at least one block in the above list which is a G-cell block, and by definintion, all the blocks in the list are G-cell blocks. Call the vertices of G_1, $\{v_1, v_2, \ldots, v_l\}$. Define

$$P_{i,j} = \{u \in G_i : (u, v) \in D \text{ for some } v \in H_j\}.$$

That is, $P_{i,j}$ is the projection of the vertices of D in block $B_{i,j}$ onto G.

We call the following procedure the *re-partitioning argument*, which we apply for part G_1 of the partition.

Notice that for any $v \in G_1$, if $v \notin P_{1,1}$, since $B_{1,1}$ is a G-cell block, there exists some vertex $u \in P_{i,1}$, for $i \geq 2$. Furthermore, such vertices u can be chosen distinctly for every v, and so we can define an injective function $f_{B_{1,1}} : \{v \in G_1 : v \notin P_{1,1}\} \to V(G)\backslash G_1$ so that $f_{B_{1,1}}(v) = u$ for v and u as defined above.

We re-partition G by exchanging every vertex $v \in G_1$ such that $v \notin P_{1,1}$ with $f_{B_{1,1}}(v)$, and calling the new set of vertices $\hat{G}_1^1$. Call the remaining sets of the partition $\hat{G}_2^1, \ldots, \hat{G}_k^1$. Next, for every i, $2 \leq i \leq k$, we remove vertices of G_1^1 which are not in $P_{1,1}$ and add them arbitrarily to other parts, remove $P_{i,1}$ from $\hat{G}_i^1$ to form G_i^1, and append G_1^1 by $P_{i,1}$ to define the vertex partition $G_1^1 = \hat{G}_1^1 \cup (\cup_{i=2}^k P_{i,1}), G_2^1, \ldots, G_k^1$. We call the blocks of this partition $B_{i,j}^1$ for $1 \leq i \leq k$ and $1 \leq j \leq l$.

We note that

(1) The new block $B_{1,1}^1$ is a G-cell block and contains at least $\gamma(G)$ vertices of D.
(2) The new blocks $B_{i,1}^1$, for $2 \leq i \leq k$, contain no vertices of D.
(3) Some or all of the new blocks $B_{i,1}^1$, for $2 \leq i \leq k$, may be empty.

If all blocks $B_{i,1}^1$ are empty for $2 \leq i \leq k$, then G_1^1 contains all the vertices of G and for every vertex $v \in G$, there is a vertex of D in $\{v\}\square H$. Since $|G| \geq \gamma(G)\gamma(H)$, the conjecture holds.

Next, if no block $\{B_{2,2}^1, B_{3,2}^1, \ldots, B_{k,2}^1\}$ is a G-cell block, then each is an H-cell block and each block $B_{i,2}^1$ contains at least $\left|G_i^1\right|$ vertices of D, for $2 \leq i \leq k$. Since the vertices of $D \cap B_{1,1}^1$ do not appear among these, and $B_{1,1}^1$ contains at least $\left|G_1^1\right|$ vertices of D, we count at least $|G| \geq \gamma(G)\gamma(H)$ vertices of D. This leaves us with the case when $B_{2,2}^1$ is a G-cell block.

We repeat the previous re-partitioning argument for the part G_2^1 without altering G_1^1. Define an injective function $f_{B_{2,2}} : \{v \in V(G_2^1) : v \notin P_{2,2}\} \rightarrow V(G)\backslash(G_1^1 \cup G_2^1)$ so that $f_{B_{2,2}}(v) = u$ for $v \in G_2^1$, $v \notin P_{2,2}$, and $u \in P_{i,2}$, for $i \geq 3$.

We exchange every vertex $v \in G_2^1$ such that $v \notin P_{2,2}$, with $f_{B_{2,2}^1}(v)$, and call the new set of vertices $\hat{G}_2^2$. Call the remaining new sets of the partition $\hat{G}_3^2, \ldots, \hat{G}_k^2$. Next, for every i, $3 \leq i \leq k$, we remove vertices of G_2^2 which are not in $P_{2,2}$ and add them arbitrarily to parts other than G_1^2, we remove $P_{i,2}$ from $\hat{G}_i^2$ to form G_i^2, and append G_1^2 by $P_{i,2}$ to define the vertex partition $G_1^2 = \hat{G}_1^2 \cup (\cup_{i=3}^k P_{i,2}), G_3^2, \ldots, G_k^2$. We call the blocks of this partition $B_{i,j}^2$ for $1 \leq i \leq k$ and $1 \leq j \leq l$.

We note that

(1) The block $B_{2,2}^2$ is a G-cell block and $B_{1,2}^2 \cup B_{2,2}^2$ contain at least $\gamma(G)$ vertices of D.
(2) The new blocks $B_{i,2}^2$, for $3 \leq i \leq k$, contain no vertices of D.
(3) Some or all of the new blocks $B_{i,2}^2$, for $3 \leq i \leq k$, may be empty.

If all blocks $B_{i,2}^2$ are empty for $3 \leq i \leq k$, then $G_1^2 \cup G_2^2$ contains all the vertices of G and for every vertex $v \in G$, there is a vertex of D in $\{v\}\square H$. Since $|G| \geq \gamma(G)\gamma(H)$, the conjecture holds.

Again, if no block $\{B_{3,3}^2, B_{4,3}^2, \ldots, B_{k,3}^2\}$ is a G-cell block, then each is an H-cell block and each block $B_{i,3}^2$ contains at least $\left|G_i^2\right|$ vertices of D, for $3 \leq i \leq k$. Since the vertices of $D \cap B_{1,1}^2$ and $D \cap (B_{1,2}^2 \cup B_{2,2}^2)$ do not appear among these, and they contain at least $\left|G_1^2\right|$ and $\left|G_2^2\right|$ vertices of D

respectively, we count at least $|G| \geq \gamma(G)\gamma(H)$ vertices of D. This leaves us with the case when $B^1_{2,2}$ is a G-cell block.

We continue re-partitioning for every set G^{i-1}_i for $3 \leq i \leq l-1$ so that

(1) The block $B^i_{i,i}$ is a G-cell block and $B^i_{1,i} \cup B^i_{2,i} \cup \cdots \cup B^i_{i,i}$ contain at least $\gamma(G)$ vertices of D.
(2) The new blocks $B^i_{j,i}$, for $i+1 \leq j \leq k$, contain no vertices of D.
(3) Some or all of the new blocks $B^i_{j,i}$, for $i+1 \leq j \leq k$, may be empty.

Suppose the re-partitioning algorithm terminates for some $i = m$. Summing the number of vertices of D in the blocks

$$\begin{array}{l} B^m_{1,1} \\ B^m_{1,2} \cup B^m_{2,2} \\ \vdots \\ B^m_{1,m} \cup \cdots \cup B^m_{m,m} \end{array}$$

produces at least $\gamma(G)\gamma(H)$ vertices.

□

For the probabilistic result, we use the Erdős-Rényi random graph model [2], $G_{n,p}$, where a graph contains n vertices and each pair of vertices is joined by an edge with probability p. Dryer [6] showed

Lemma 2.3. [6] *Choose $p \in [0,1)$ and T any vertex set of size $(1+\epsilon)\log_q n$ in $G_{n,p}$, where $\epsilon > 0$ and $q = \frac{1}{1-p}$. Then $Pr(T$ is a dominating set$)$ approaches 1 as n approaches infinity.*

Applying Dryer's result to the condition $|G| \geq \gamma(G)\gamma(H)$ and $|H| \geq \gamma(G)\gamma(H)$ produces

Corollary 2.4. *Vizing's conjecture holds for almost all pairs of graphs G, H with $|G| \geq |H|$, satisfying $|G| \leq q^{\frac{|H|}{\log_q |H|}}$ for $q = \frac{1}{1-p}$ and p the edge probability of the Erdős-Rényi random graph.*

It would be interesting to prove the following

Conjecture 2.5. *Vizing's conjecture holds for almost all pairs of graphs.*

References

[1] A. M. Barcalkin and L. F. German, *The external stability number of the Cartesian product of graphs*, Bul. Akad. Stiinte RSS Moldoven, Vol. 1, p. 5-8, 94, 1979.
[2] B. Bollobás, *Random Graphs*, 2nd Edition, 2001, Cambridge University Press.
[3] B. Brešar, P. Dorbec, W. Goddard, B. Hartnell, M. Henning, S. Klavžar, D. Rall, *Vizing's Conjecture: A survey of Recent Results*, Journal of Graph Theory, DOI: 10.1002/jgt.20565, Vol. 69 (1), p. 46-76, 2012.
[4] B. Brešar, D. Rall, *Fair reception and Vizing's conjecture*, Journal of Graph Theory, Vol. 61, p. 45-54, 2009.
[5] R. Diestel, *Graph Theory, Third Edition*, Springer-Verlag, Heidelberg Graduate Texts in Mathematics, Vol. 173, New York, 2005.
[6] P. A. Dryer, *Applications and Variations of Domination in graphs, PhD Dissertation, Department of Mathematics, Rutgers University*
[7] W. Imrich, S. Klavžar, *Product Graphs: Structure and Recognition*, Wiley, ISBN 0-471-37039-8, 2000.

[8] S. Suen, J. Tarr, *An Improved Inequality Related to Vizing's Conjecture*, Electronic Journal of Combinatorics, Vol. 19 (1), 2012, #P8.

[9] V. G. Vizing, *The Cartesian Product of Graphs*, Vycisl. Sistemy, Vol. 9, p. 30-43, 1963

Aziz Contractor (`acontractor@student.clayton.edu`)

Elliot Krop (`elliotkrop@clayton.edu`)

Department of Mathematics, Clayton State University